ФІЗИКО-МАТЕМАТИЧНІ НАУКИ

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USING LAPLACE TRANSFORMS FOR ELECTRICAL CIRCUITS

1. The Laplace Transform

1.1. Laplace Transform. Inverse Transform

If $f(t)$ is a function defined for all $t \geq 0$, its Laplace transform is the integral of $f(t)$ times e^{-st} from $t = 0$ to ∞ It is a function of s, say, $F(s)$ and is denoted by $L(x)$;thus

$$F(s) = L(f) = \int_0^\infty e^{-st} f(t)dt \tag{1}$$

Here we must assume that is $f(t)$such that the integral exists. This assumption is usually satisfied in applications-we shall discuss this near the end of the section.

Not only is the result called $F(s)$ the Laplace transform, but the operation just described, which yields from $F(s)$ a given, is $f(t)$ also called the Laplace transform. It is an «integral transform»

$$F(s) = \int_0^\infty k(s,t) f(t)dt \tag{2}$$

With «kernel» $k(s,t) = e^{-st}$

Furthermore, the given function $f(t)$ in (1) is called the inverse transform of and $F(s)$ is denoted by $L^{-1}(F)$; that is, we shall write

$$f(t) = L^{-1}(F) \tag{3}$$

Notation: Original functions depend on *t* and their transforms on s-keep this in mind! Original functions are denoted by *lowercase letters* and their transforms by the same *letters in capital,* so that $F(s)$ denotes the transform of $f(t)$, and $Y(s)$ denotes the transform of $y(t)$, and so on.

EXAMPLE 1 Laplace Transform

Let $f(t) = 1$ when $t \geq 0$ Find $F(s)$.

Solution: From (1) we obtain by integration

$$L(f) = L(1) = \int_0^\infty e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s} \ (s > 0)$$

Our notation is convenient, but we should say a word about it. The interval of integration in (1) is infinite. Such an integral is called an improper integral and, by definition, is evaluated according to the rule

$$\int_0^\infty e^{-st} f(t)dt = \lim_{T-\infty} \int_0^T e^{-st} f(t)dt$$

Hence our convenient notation means

$$\int_0^\infty e^{-st}\, dt = \lim_{T-\infty} \left[-\frac{1}{s} e^{-st}\right] = \lim_{T-\infty} \left[-\frac{1}{s} e^{-sT} + \frac{1}{s} e^0\right] = \frac{1}{s} \ (s > 0)$$

1.2. Transforms of Derivatives and Integrals. ODEs

The Laplace transform is a method of solving ODEs and initial value problems. The crucial idea is that operations of calculus on functions are replaced by operations of algebra on transforms. Roughly, differentiation of $f(t)$ will correspond to multiplication of $L(f)$ by s (see Theorems 1) and integration of $f(t)$ to division of $L(f)$ by s. To solve ODEs, we must first consider the Laplace transform of derivatives

THEOREM 1 Laplace Transform of Derivatives

The transforms of the first and second derivatives of $f(t)$ satisfy

$$L(f') = sL(f) - f(0) \tag{4}$$

$$L(f'') = s^2L(f) - sf(0) - f'(0) \tag{5}$$

Formula (4) holds if $f(t)$ is contilluous for all $t \geq 0$ and satisfies the growth restriction (5) in Sec. 1.1 and $f'(t)$ is piecewise continuous on every finite interval on the semi-axis $t \geq 0$. Similarly, (4) holds if f and f' are continuous for $t \geq 0$ all and satisfy the growth restriction and f» is piecewise continuous on every finite interval on the semi-axis $t \geq 0$

1.3. Differential Equations, Initial Value Problems

We shall now discuss how the Laplace transform method solves ODEs and initial value problems. We consider an initial value problem

$$y'' + ay' + by = r(t)\ y(0) = K_0\ \ y'(0) = K_1 \tag{6}$$

Where a and b are constant. Here $r(t)$ is the given input (driving force) applied to the mechanical or electrical system and $y(t)$ is the output (response to the input) to be obtained. In Laplace's method we do three steps:

Step 1: Setting up the subsidiary equation.

This is an algebraic equation for the transform $Y = L(y)$ obtained by transforming (4) by means of (1) and (4), namely,

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s) \tag{7}$$

Where $R(s) = L(r)$. Collecting the Y-terms, we have the subsidiary equation

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s) \tag{8}$$

Step 2: Solution of the subsidiary equation by algebra.

We divide by $s^2 + as + b$ and use the so-called transfer function

$$Q(s) = \frac{1}{(s^2+as+b)} = \frac{1}{\left(s+\frac{1}{2}a\right)^2+b-\frac{1}{4}a^2} \tag{9}$$

This gives the solution

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s) \tag{10}$$

If $y(0) = y'(0) = 0, this\ is\ simply\ Y = RQ, hense$

$$Q = \frac{Y}{R} = \frac{L(output)}{L(input)}.$$

And this explains the name of Q. Note that Q depends neither on $r(t)$ nor on the initial conditions.

Step 3. III version of Y to obtain $\boldsymbol{y = L^{-1}(Y)}$.

We reduce (10) to a sum of terms whose inverses can be found from the tables or by a CAS, so that we obtain the solution

$$y(t) = L^{-1}(y) \text{ of.} \tag{11}$$

EXAMPLE 2 Comparisons with the Usual Method

Solve the initial value problem

$$y'' + y' + 9y = 0, y(0) = 0.16\, y'(0) = 0$$

Solution. From (1) and (6) we see that the subsidiary equation is

$$s^2Y - 0.16s + sY - 0.16 + 9Y = 0, thus\ (\ s^2 + s + 9)Y = 0.16(s + 1)$$

The solution is

$$Y = \frac{0.16(s+1)}{s^2 + s + 9} = \frac{0.16(s + 1/2)}{(s + 1/2)^2 + 35/4}$$

Hence by the first shifting theorem and the formulas for cos and sin in Table 1 we obtain

$$y(t) = L^{-1}(Y) = e^{-\frac{t}{2}}\left(0.16cos\sqrt{\frac{35}{4}}t + \frac{0.08}{\frac{1}{2}\sqrt{35}}sin\sqrt{\frac{35}{4}}t\right)$$

$$= e^{-0.5t}(0.16\cos 2.96t + 0.027sin2.96t)$$

Advantages of the Laplace Method

a) Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE. See Example 3

b) Initial values are automatically taken care of See Examples 3 and 4.

c) Complicated inputs $r(t)$ can be handled very efficiently, as we show in the next sections.

2. Electrical Circuits

We examine a number of basic circuits, solve the problem analytically using the Laplace transform, and then check our result empirically against an actual circuit.

EXAMPLE 3 Responses of an RC-Circuit to a Single Rectangular Wave

Find the current $i(t)$ in the RC-circuit in Fig. 2 if a single rectangular wave with voltage V_0 is applied. The circuit is assumed to be quiescent before the wave is applied.

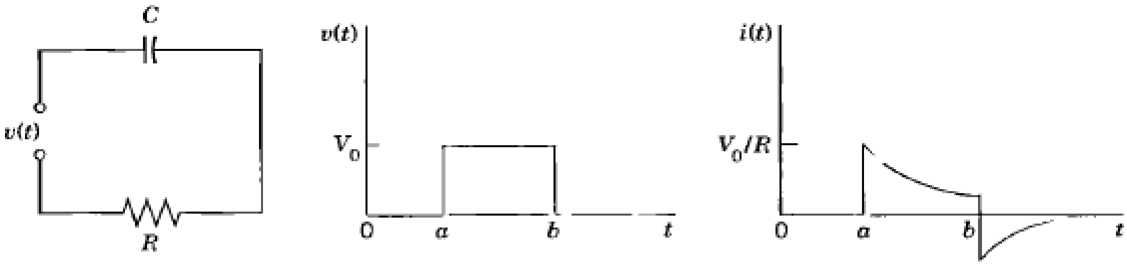

Figure 2. RC-circuit, electromotive force $v(t)$, and current

Solution: The input is $V_0[u(t - a) - u(t - b)]$. Hence the circuit is modeled by the integro-differential equation

$$Ri(t) + \frac{q(t)}{C} = Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t) = V_0[u(t-a) - u(t-b)].$$

We obtain the subsidiary equation

$$RI(s) + \frac{I(s)}{sC} = \frac{V_0}{s}[e^{-as} - e^{-bs}]$$

Solving this equation algebmically for $I(s)$. We get

$$I(s) = F(s)[e^{-as} - e^{-bs}];\ F(s) = \frac{V_0/R}{s + 1/(RC)}\ ;\ L^{-1}(F) = \frac{V_0}{R}e^{-t/(RC)}$$

the last expression being obtained from Table 1. Hence Theorem 1 yields the solution (Figure 2)

$$i(t) = L^{-1}(I) = L^{-1}\{e^{-as}F(s)\} - L^{-1}\{e^{-bs}F(s)\} = \frac{V_0}{R}\left[e^{-\frac{t-a}{RC}} - e^{-\frac{t-b}{RC}}u(t-b)\right];$$

That is $i(t) = 0$ if $t < a$, and

$$i(t) = \begin{Bmatrix} K_1 e^{-\frac{t}{RC}}\, if\ a < t < b \\ (K_1 - K_2)e^{-\frac{t}{RC}}\, if\ a > b \end{Bmatrix}$$

Where $K_1 = \frac{V_0 e^{\frac{a}{RC}}}{R}$ $and\ K_2 = V_0\frac{e^{\frac{b}{RC}}}{R}$

CONCLUSIONS

- The main purpose of Laplace transforms is the solution of differential equations and systems of such equations, as well as corresponding initial value problems. The Laplace transform $F(s) = L(f)$ of a function $f(t)$ is defined by

$$F(s) = L(f) = \int_0^\infty e^{-st}f(t)dt \text{ (Sec. 1.1)}$$

- This definition is motivated by the property that the differentiation of f with respect to t corresponds to the multiplication of the transform F by s; more precisely,

$$L(f') = sL(f) - f(0) \text{ (sec.1.2)}$$

$$L(f'') = s^2L(f) - sf(0) - f'(0)$$

etc. Hence by taking the transform of a given differential equation, *(a, b* constant)

$$y» + ay' + by = r(t) \text{ (sec.1.3)}$$

and writing $L(y) = Y(s)$,

- we obtain the subsidiary equation

$$(s^2 + as + b)Y = L(r) + sf(0) + f'(0) + af(0) \text{ (sec.1.3)}$$

- The Laplace transform is a useful tool for solving differential equations analytically, especially in cases with discontinuous forcing terms or a periodic, non-sinusoidal forcing term.
- In addition, analysis of circuits in the s-domain can yield insights into the frequency response of the circuits.

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