ФІЗИКО-МАТЕМАТИЧНІ НАУКИ

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USING LAPLACE TRANSFORM OF THE HEAVISIDE AND DIRAC DELTA FUNCTIONS

1. The Laplace Transform

1.1. Laplace Transform. Inverse Transform.

If $f\ (t)$ is a function defined for all $t \geq 0$, its Laplace transform is the integral of $f\ (t)$ times e^{-st} from $t = 0$ to ∞ It is a function of s, say, $F\ (s)$ and is denoted by $L\ (x)$, thus

$$F(s) = L(f) = \int_0^\infty e^{-st} f(t) dt \tag{1}$$

Here we must assume that is $f\ (t)$such that the integral exists. This assumption is usually satisfied in applications-we shall discuss this near the end of the section.

Not only is the result called $F\ (s)$ the Laplace transform, but the operation just described, which yields from $F\ (s)$ a given, is $f\ (t)$ also called the Laplace transform. It is an «integral transform»

$$F(s) = \int_0^\infty k(s,t) f(t) dt \tag{2}$$

With «kernel» $k(s,t) = e^{-st}$

Furthermore, the given function $f\ (t)$ in (1) is called the inverse transform of and $F\ (s)$ is denoted by $L^{-1}\ (F)$; that is, we shall write

$$f\ (t) = L^{-1}\ (F) \tag{3}$$

Notation: Original functions depend on t and their transforms on s-keep this in mind! Original functions are denoted by *lowercase letters* and their transforms by the same *letters in capital*, so that $F\ (s)$ denotes the transform of $f\ (t)$, and $Y\ (s)$ denotes the transform of $y\ (t)$, and so on.

1.2. Transforms of Derivatives and Integrals. ODEs

The Laplace transform is a method of solving ODEs and initial value problems. The crucial idea is that operations of calculus on functions are replaced by operations of algebra on transforms. Roughly, differentiation of $f\ (t)$ will correspond to multiplication of $L\ (f)$ by s (see Theorems 1) and integration of $f\ (t)$ to division of $L\ (f)$ by s. To solve ODEs, we must first consider the Laplace transform of derivatives

THEOREM 1: The transforms of the first and second derivatives of $f\ (t)$ satisfy

$$L(f') = sL(f) - f\ (0) \tag{4}$$

$$L\ (f'') = s^2 L(f) - sf(0) - f'\ (0) \tag{5}$$

Formula (4) holds if $f\ (t)$ is contilluous for all $t\ \geq 0$ and satisfies the growth restriction (5) in Sec. 1.1 and $f'\ (t)$ is piecewise continuous on every finite interval on the semi-axis $t\ \geq 0$. Similarly, (4) holds if $f\ and$ f' are continuous for $t\ \geq 0$ all and satisfy the growth restriction and f» is piecewise continuous on every finite interval on the semi-axis $t \geq 0$.

2. Heaviside Step function and Dirac Delta

2.1. Heaviside Step function

The unit step function or Heaviside function. The unit step function or Heaviside function is defined by

$$u(x) = \begin{cases} 1 \; for \; x \geq 0 \\ 0 \; for \; x < 0 \end{cases}$$

The most often-used formula involving Heaviside function is the characteristic function of the interval $a \leq t < b$, given by

$$u(t-a) - u(t-b) = \begin{cases} 1 \; a \leq t < b \\ 0 \; t < a, t \geq b \end{cases}$$

To illustrate, a square wave $sqw(t) = (-1)^{floor(t)}$ can be written in the series from

$$\sum_{n=0}^{\infty} (-1)^n \big(u(t-n) - u(t-n-1)\big).$$

2.2. Dirac Delta function

To model situations of that type, we consider the function

$$f_k(t-a) = \left\{ \begin{matrix} \frac{1}{k} \; if \; a \leq t \leq a+k \; Figure.1 \\ 0 \; otherwise \end{matrix} \right\} \tag{6}$$

This function represents. For instance. A force of magnitude $1/k$ acting from $t = a$ to $t = a + k$, where k is positive and small. In mechanics, the integral of force acting over a time interval $a \leq t \leq a + k$ is called the impulse of the force; similarly for electromotive forces $E\ (t)$ acting on circuits. Since the blue rectangle in Figure 1 has area 1, the impulse of f_k in (6) is

$$I_k = \int_0^{\infty} f_k(t-a)dt = \int_0^{a+k} \frac{1}{k} dt = 1 \tag{7}$$

To find out what will happen if k becomes smaller and smaller, we take the limit of f_k as $k -> 0$ $(k > 0)$. this limit is denoted by $\delta\ (t - a)$, that is,

$$\delta(t - a) = \lim_{k \to 0} f_k\ (t-a)$$

$\delta\ (t - a)$ is called the Dirac delta function or the unit impulse function.

$\delta\ (t - a)$ is not a function in the ordinary sense as used in calculus, but a so-called generalizedjullction. To see this, we note that the impulse l_k of f_k is I, so that from (6) and (7) by taking the limit as $k -> 0$ we obtain

$$\delta(t - a) = \left\{ \begin{matrix} \infty \; if \; t = a \\ 0 \; otserwise \end{matrix} \; and \; \int_0^{\infty} \delta(t - a)dt = 1 \right\}$$

but from calculus we know that a function which is everywhere 0 except at a single point must have the integral equal to O. Nevertheless, in impulse problems it is convenient to operate on $\delta(t - a)$ as though it were an ordinary function. In particular, for a *continuous* function $g\ (t)$ one uses the property

$$\int_0^\infty g(t)\delta(t - a)dt = ga$$

Which is plausible by (7)? To obtain the Laplace transform of $\delta(t - a)$, we write

$$f_k(t-a) = \frac{1}{k}\ [u(t-a) - u\ (t-\ (a+k))]$$

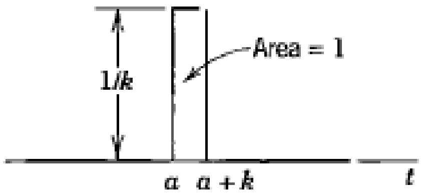

Fig. 1. The function $\mathbf{f_k}$ $(\mathbf{t} - \mathbf{0})$ in (6)

EXAMPLE 1: Unrepeated Complex Factors. Damped Forced Vibrations

Solve the initial value problem for a damped mass-spring system acted upon by a sinusoidal force for some time interval,

$$y'' + 2y' + 2y = r(t), r(t)$$

$$= 10\sin 2t\ if\ 0 < t < \pi\ and\ 0\ if\ t > \pi;\ y(0) = 1\ y'(0) = -5$$

Solution. The subsidiary equation

$$(s^2Y - s + 5) + 2(sY - 1) + 2Y = 10\frac{2}{s^2+4}\ (1 - e^{-\pi s})$$

We collect the Y-terms, $(s^2 + 2s + 2)Y$, take $-s + 5 - 2 = -s + 3$ to the right, and solve,

$$Y = \frac{20}{(s^2+4)\ (s^2+2s+2)} - \frac{20e^{-\pi s}}{(s^2+4)\ (s^2+2s+2)} + \frac{s-3}{(s^2+2s+2)} \qquad (8)$$

$$L^{-1}\left\{\frac{s+1-4}{(s+1)^2+1}\right\} = e^{-t}(\cos t - 4\sin t) \qquad (9)$$

In the first fraction in (8) we have unrepeated complex roots, hence a partial fraction representation

$$\frac{20}{(s^2+4)\ (s^2+2s+2)} = \frac{As+B}{s^2+4} + \frac{Ms+N}{s^2+2s+2} \qquad (10)$$

Multiplication by the common denominator gives

$$20 = (As + B)\ (s^2 + 2s + 2) + (Ms + N)\ (s^2 + 4).$$

We determine A, B, M, N. Equating the coefficients of each power of s on both sides gives the four equations

(a) $[s^3]$: $0 = A + M$ (b) $[s^2]$: $0 = 2A + B + N$

(c) $[s^1]$: $0 = 2A + 2B + 4M$ (d) $[s^0]$: $20 = 2B + 4N.$

We can solve this, for instance, obtaining $M = -A$ from (a), then $A = B$ from (c), then $N = -3A$ from (b), and finally $A = -2$ from (d). Hence $A = -2$,

$B = -2, M = 2, N = 6$. And the Fust fraction in (8) has the Represent ation

$$\frac{-2s-2}{s^2+4}+\frac{2(s+1)+6-2}{(s+1)^2+1} \tag{11}$$

Inverse transform $-2\,cos\,2t - \,sin\,2t + e^{-t}\,(2cos\,t + 4\,sin\,t)$
The sum of this and is the, solution of the problem for $0 < f < \pi$ namely.

$$y(t) = 3e^{-t}\,\cos t - 2\cos 2t - \sin t \;\; if\;(0 < t < \pi) \tag{12}$$

In the second fraction in (11) taken with the minus sign we have the factor $e^{-\pi s}$ so that from (11) and the second shifting theorem we get the inverse transform

$$+2\cos(2t-2\pi)+\sin(2t-2\pi)-e^{-\,(t-\pi)}[2\cos(t-\pi)+4\sin(t-\pi)]$$

$$= 2\cos 2t + \sin 2t + e^{-(t-\pi)}\,(2\cos t + 4\sin t)$$

Sum of the this and (12) is the solution for $> \pi$,

$$y(t) = e^{-t}[(3+2e^{\pi})\cos t + 4e^{\pi}\sin t] \tag{13}$$

Figure 2 shows (12) (for $0 < t < \pi$) and (13) (for $t > \pi$), a beginning vibration, which goes to zero rapidly because of the damping and the absence of a driving force after $t = \pi$

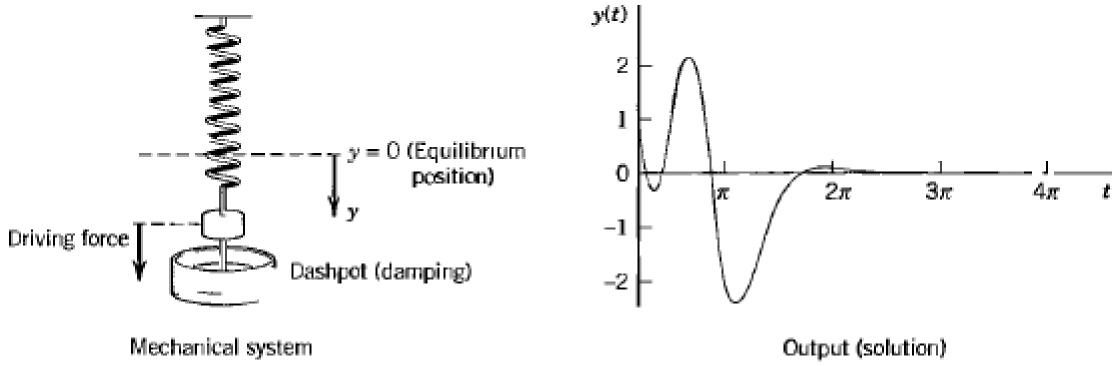

Fig. 2. Damped Forced Vibrations

CONCLUSIONS

• The main purpose of Laplace transforms is the solution of differential equations and systems of such equations, as well as corresponding initial value problems. The Laplace transform $F\,(s) = L\,(f)$ of a function $f\,(t)$ is defined by

$$F(s) = L(f) = \int_0^{\infty} e^{-st} f\,(t)dt \text{ (sec. 1.1)}$$

• This definition is motivated by the property that the differentiation of f with respect to t corresponds to the multiplication of the transform F by s; more precisely,

$$L(f') = sL(f) - f\,(0) \text{ (sec. 1.2)}$$

$$L(f'') = s^2L(f) - sf(0) - f'(0)$$

• The reason is that we shall introduce two auxiliary functions, the unit step function or Heaviside function $u\,(t - a)$ and Dirac's delta $\delta\,(t - a)$.

• Delta function is one of so-called generalized functions, which are not functions in ordinary sense but as an operator that sometimes can be represented by ordinary functions.

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НОВИЙ ВАРІАНТ МЕТОДУ ДОСЛІДЖЕННЯ МЕХАНІЧНИХ ХАРАКТЕРИСТИК ПЛАСТИН ТА ОБОЛОНОК ЗМІННОЇ ТОВЩИНИ ПІДВИЩЕНОЇ ТОЧНОСТІ

Розвиток сучасної промисловості неможливо представити без використання оболонкових конструкцій, як результат, виникають різноманітні форми об'єктів, які знаходять свій широкий спектр застосування у різних областях науки та техніки. Прикладом таких конструкцій є: фланці, дискові пружини, сильфони, котли, балони, ротори, барабани, трубопроводи, корпуса (літаків, вертольотів, ракет, кораблів, ядерних реакторів). Беручи до уваги практичну значущість кожного елемента із перерахованих, стає зрозуміло, що до оболонкових конструкцій висуваються жорсткі умови, розрахунок яких пов'язаний із побудовою розрахункових схем та математичних моделей із застосуванням сучасних чисельних методів досліджень, які можна реалізувати в пакетах програм та програмних комплексів [2].

Значна увага до оболонок та необхідність створення власноруч різноманітних оболонкових конструкцій пояснюється тим, що вони володіють надважливими властивостями на міцність, жорсткість, стійкість.

Класичним підходом до побудови теорії оболонок є використання гіпотез або спрощуючи пропозицій, перевагою використання такого підходу є те, що вихідні співвідношення мають достатньо просте математичне формулювання, у такий спосіб можна звести вихідні співвідношення тривимірної теорії пружності до двовимірних рівнянь. Наукові праці присвячені теорії оболонок із застосуванням спрощуючих гіпотез, належать С. П. Тимошенку, І. Г. Бубнову, Б. Г. Галеркіну.

При розробці методів обчислень оболонок із сучасних композитних матеріалів, для яких характерна анізотропія та неоднорідність механічних властивостей оболонок на які поширюються локальні впливи, необхідно враховувати поперечні деформації і напруження, які не розглядає класична теорія. У цьому випадку розробляють уточнену теорію оболонок, яка дозволяє