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UDC 517

SYMMETRY AND EXACT SOLUTIONS OF CYLINDRICALLY SYMMETRIC WAVE EQUATION

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The article is describes the search of exact solutions of nonlinear differential equations by the method of symmetry reduction Sophus Lie. The article deals with theoretical information and features of the method Lie. The method Lie applied to a cylindrically symmetric nonlinear wave equation D'Alembert. Sets out the conditions conformal invariance of this equation. Used conformal symmetry of the equations for find exact solutions. Found some solutions of the reduced equations and exact solutions of the cylindrically symmetric wave equation.
Keywords: method of Lie, symmetrical analysis, cylindrically symmetric wave equation, D'Alembert equation, symmetry reduction, algebraic invariance.

Introduction. The result of searches the decisions of various problems mathematical description of natural processes is differential equations. Development of integration methods differential equations assisted to becoming mathematical physics science. In the development process, widespread enough approach of construction exact decisions of nonlinear differential equations was the principle of symmetry equations that started by Norwegian mathematician Sophus Lie. Symmetry in mathematical and theoretical physics is considered as the principle from a multiple number of possible equations and models the relationship of the actual processes to select the most comfortable to use. This principle plays an important role in modern scientific researches. All basic equalizations of mathematical physics – Newton, D'Alambert, Laplace, Maxwell, Schrodinger etc. – own wide symmetry properties. The method of S. Lie based on application algebra invariance of differential equalization for finding him exact decisions is symmetries of Lie. The fundamental opening of S. Lie consists in that – difficult nonlinear conditions of invariance differential equalizations, are in the case of continuous groups, it is possible to replace equivalent, but more simple linear conditions, which represent the «infinitesimal invariance» of differential equalizations relatively of constituents this group. The most general group of continuous symmetries of the system can be certain obviously. An important contribution to this research done Wilhelm Fushchych. An important area, in which he and his students worked, were use algebra and groups the invariance of nonlinear differential equations to find their solutions. For search solutions we using sub algebras, algebra invariance of differential equations build special substitution, is "ansatzes". Ansatzes reduces original equation to equation with fewer independent variables. In this way, we managed to get the whole classes of exact solutions of many basic equations of mathematical physics – Liouville, D'Alembert, Monge-Ampere, Born-Infeld, Schrodinger and many others. Presently actual researches of all basic differential equalizations of mathematical physics and their generalizations by symmetry and conditional symmetry.

1. Use of symmetry to searching precise decisions cylindrical-symmetric non-linear wave equation

It is known, that the wave equation looks as

$$\square u = F(x,u), \tag{1}$$

If $F(x,u) \neq 0$, equation (1) retains conformal symmetry $AC(1,n)$ only in case $F(x,u) = \lambda u^{\frac{n+3}{n-1}}, n \neq 1$, where $\lambda -$ arbitrary constant. During the description of actual physical processes, that is when $n = 3$, we use equation

$$u_{00} - u_{11} - u_{22} - u_{33} = F(u). \tag{2}$$

Let process, which is describe by equation (2), is cylindrically symmetric. It is means that

$$u(x_0, x_1, x_2, x_3) = u(x_0, x_1, \rho), \tag{3}$$

where $\rho = \sqrt{x_2^2 + x_3^2}$. Substitute (3) in (2), we get

$$u_{00} - u_{11} - u_{\rho\rho} - \frac{1}{\rho}u_\rho = F(u).$$

If rewritten told above for the arbitraries number of explanatory variables $u = u(y_0, y_1, ..., y_{n+N})$.

If this process which is describe by equation

$$u_{00} - u_{11} - ... - u_{n+Nn+N} = F(u),$$

has a spherical symmetry, that is $u = u(y_0, y_1, ..., y_{n-1}, \rho)$, where $\rho = \sqrt{y_n^2 + y_{n+N}^2}$, we will similarly receive the equation

$$u_{00} - u_{11} - ... - u_{n-1n-1} - u_{\rho\rho} - \frac{N}{\rho}u_\rho = F(u). \tag{4}$$

Let us assume $y_0 = x_0, y_1 = x_1, ..., y_{n-1} = x_{n-1}, \rho = x_n$, then equation (4) looks as

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$$u_{00} - u_{11} - \ldots - u_{nn} - \frac{N}{x_n}u_\rho = F(u). \qquad (5)$$

where $u = u(x), x = x(x_0, \bar{x}) \in R_{1+n}$. Rewritten equation (5) as follows

$$\Box u - \frac{N}{x_n}u_n = F(u). \qquad (6)$$

Equation (6) has conformal symmetry. Use the theorem.

Theorem. Equation (6) if $N \neq 0$ invariant relatively conformal algebra $AC(1, n-1)$:

$$< \partial_\alpha, J_{\alpha\beta} = x^\alpha\partial_\beta - x^\beta\partial_\alpha,\ D = x_\alpha\partial_\alpha + x_n\partial_n - \frac{1-n-N}{2}u\partial_u,$$

$$K_\alpha = 2x^\alpha D - (x_\beta x^\beta - x_n^2)\partial_\alpha >,\ \alpha, \beta = \overline{0, n-1},$$

in only case when,

$$F(u) = \lambda u^k,\ N = 1 - n + \frac{4}{k-1}, \qquad (7)$$

where λ i k – arbitrary constant.

In a case $n = 2$ and under a condition (7) equation (6) has an appearance

$$u_{00} - u_{11} - u_{22} - \frac{N}{x_2}u_2 = \lambda u^k,\ N = \frac{5-k}{k-1}, N \neq 0, k \neq 1. \qquad (8)$$

Use a symmetry of the equation (8) for finding solutions, which we will finding in same an appearance as

$$u(x) = f(x)\varphi(\omega^1, \omega^2), \qquad (9)$$

In a formula (9) φ – unknown function which needs to be defined, and $\omega^0 = \frac{u(x)}{f(x)}, \omega^1 = \omega^1(x), \omega^2 = \omega^2(x)$ – is invariant conformal algebra $AC(1,1)$. If substituted ansatz (9) in equation (8), we will have

$$\begin{aligned}&\omega^1_\mu\omega^{1,\mu}\varphi_{11} + 2\omega^1_\mu\omega^{2,\mu}\varphi_{12} + \omega^2_\mu\omega^{2,\mu}\varphi_{22} + \\ &+\left(\Box\omega^1 - \frac{5-k}{(k-1)x_2}\omega^2_2 + \frac{2}{f}f_\mu\omega^{1,\mu}\right)\varphi_1 + \\ &+\left(\Box\omega^2 - \frac{5-k}{(k-1)x_2}\omega^2_2 + \frac{2}{f}f_\mu\omega^{2,\mu}\right)\varphi_2 + \\ &+\frac{1}{f}\left(\Box f - \frac{5-k}{(k-1)x_2}f_2\varphi\right) - \lambda f^k\varphi^k = 0.\end{aligned} \qquad (10)$$

Having considered a formula (10) it is compatible to table 1.1, where the corresponding specified values of invariant variables, we will receive nine not equivalent reduced equations for function definition φ:

1°. $\varphi_{11} + \frac{5-k}{(k-1)\omega^1}\varphi_1 + \lambda\varphi^k = 0,$

2°. $\varphi_{11} - 2\omega^1\varphi_{12} - 4\omega^2\varphi_{22} + \frac{5-k}{(k-1)\omega^1}\varphi_1 - 4\frac{k+1}{k-1}\varphi_2 + \lambda\varphi^k = 0,$

3°. $((\omega^1)^2 + 1)\varphi_{11} + \omega^1\omega^2\varphi_{12} + ((\omega^2)^2 + 4)\varphi_{22} + 3\omega^1\varphi_1 + 3\omega^2\varphi_2 + q(\varphi) = 0,$

4°. $(\omega^1)^2\varphi_{11} + \omega^1\omega^2\varphi_{12} + ((\omega^2)^2 + 4)\varphi_{22} + 3\omega^1\varphi_1 + 3\omega^2\varphi_2 + q(\varphi) = 0,$

Table 1.1.

Invariants conformal algebra $AC(1,1)$

№	ω_0	ω_1	ω_2
1	u	x_2	αx_2
2	u	x_2	x_2
3	$ux_2^{\frac{N+1}{2}}$	$\frac{bx}{x_2}$	$\frac{x^2+1}{x_2}$
4	$ux_2^{\frac{N+1}{2}}$	$\frac{\alpha x}{x_2}$	$\frac{x^2+1}{x_2}$
5	$ux_2^{\frac{N+1}{2}}$	$\frac{\alpha x}{x_2^2}$	$\beta x + m\ln x_2$
6	$ux_2^{\frac{N+1}{2}}$	$\frac{\alpha x}{x_2^3}$	$x_2\beta x$
7	$ux_2^{\frac{N+1}{2}}$	$\frac{x^2\beta x + \alpha x}{x_2^2}$	$\frac{(x^2+1)^2 + 4(bx)^2}{x_2^2}$
8	$ux_2^{\frac{N+1}{2}}$	$\mathrm{arctg}\frac{x^2-1}{2ax} - 2\mathrm{arctg}\frac{x^2+1}{2bx}$	$\frac{(x^2+1)^2 + 4(bx)^2}{x_2^2}$
9	$ux_2^{\frac{N+1}{2}}$	$\frac{x^2\beta x + \alpha x}{x_2^2}$	$\frac{1}{2}\ln\frac{(x^2)^2 + (\alpha x)^2}{x_2^2} - \mathrm{arctg}\frac{x^2}{\alpha x}$

5°. $4(\omega^1)^2\varphi_{11} - 2(1 + m\omega^1)\varphi_{12} + m^2\varphi_{22} + 8\omega^1\varphi_1 - 2m\varphi_2 + q(\varphi) = 0,$

6°. $9(\omega^1)^2\varphi_{11} - (2 + \omega^1\omega^2)\varphi_{12} + (\omega^2)^2\varphi_{22} + 15\omega^1\varphi_1 - \omega^2\varphi_2 + q(\varphi) = 0,$

7°. $((\omega^1)^2 + 1)\varphi_{11} - \omega^1(\omega^2 + 2)\varphi_{12} + \omega^2(\omega^2 + 4)\varphi_{22} + 2\omega^1\varphi_1 - (\omega^2 + 2)\varphi_2 + \frac{q(\varphi)}{4} = 0,$

8°. $\left(\frac{4}{\omega^2} + \frac{1}{\omega^2 + 4}\right)\varphi_{11} + \omega^2(\omega^2 + 4)\varphi_{22} + 2(\omega^2 + 2)\varphi_2 + \frac{q(\varphi)}{4} = 0,$

9°. $4((\omega^1)^2 + 1)\varphi_{11} + 2(\omega^1 - 1)\varphi_{12} + \varphi_{22} + 8\omega^1\varphi_1 - 2\varphi_2 + q(\varphi) = 0,$

where $q(\varphi) = 4\frac{k-2}{(k-1)^2}\varphi + \lambda\varphi^k.$

In this table the following designations are injected: $\alpha x = \alpha_0 x_0 - \alpha_1 x_1, bx = b_0 x_0 - b_1 x_1,$ $x^2 = x_0^2 - x_1^2 - x_2^2; a, b, \alpha, \beta$ – arbitrary constants vectors which satisfying conditions $a^2 = -b^2, ab = 0, \alpha = a + b, \beta = a - b, m = \text{constant}.$

2. Partial decisions of reduced equations

Having analyzed received reduced equations, we will specify partial decisions. If in 7° $\varphi_1 = 0$, let us receive an ordinary differential equation:

$$\omega^2(\omega^2 + 4)\varphi_{22} + 2(\omega^2 + 2)\varphi_2 + \frac{k-2}{(k-1)^2}\varphi + \frac{\lambda}{4}\varphi^k = 0,$$

partial solution is

$$\varphi(\omega^2) = \left[-\frac{\lambda}{16}(k-1)^2\omega^2\right]^{\frac{1}{1-k}}. \quad (11)$$

Ansatz (9) and function (11) give the chance to find the solution of the equation (6)

$$u(x) = \left[-\frac{\lambda}{16}(k-1)^2\{x^2 - x_2^2 + 1) + 4(bx)^2\}\right]^{\frac{1}{1-k}}.$$

The results received above can be multiply by means of transformation of an invariance of the equation (6). These transformations have an appearance

$$x_\alpha \longrightarrow \frac{e^m c_{\alpha\beta}(x_\beta - \theta_\beta(x^2 - x_2^2))}{\sigma},$$

$$x_2 \longrightarrow \frac{e^m x^2}{\sigma},$$

$$u \longrightarrow u\left[\frac{e^m}{\sigma}\right]^{\frac{2}{1-k}},$$

where $\sigma = 1 - 2\theta_\alpha x^\alpha + \theta_\alpha\theta^\alpha(x^2 - x_2^2), b_\alpha, c_{\alpha\beta}, \theta_\alpha, m$ – arbitrary parameters.

3. Conclusions

In the this article sets out the conditions conformal invariance cylindrically symmetric nonlinear wave equation D'Alembert $\Box u - \frac{N}{x_n}u_n = F(u)$ relatively conformal algebra $AC(1, n-1)$. This eliminated the dependence between the exponent functions $F(x,u) = \lambda u^{\frac{n+3}{n-1}}$ of the number of spatial variables exist for the classical equation $\Box u = F(x,u)$ for the requirement of conformal invariance relation algebra $AC(1,n)$. A search of invariants and ansatzes conformal algebra $AC(1,1)$, and a partial decisions of reduced equations.

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